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Diversification in Firm Valuation: A Multivariate Copula Approach*

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Abstract

We introduce a new discounted cash flow model which adopts the diversification effect of multi-business firms. We face two challenges: One is examining how different diversification extents can affect the firm value due to risk reduction, and the other is modeling segment-specific cash flows and discount rates to reflect the differences in risk and growth characteristics across the different businesses that a firm operates in. Since the co-movement of business segments depends on the state of the economy, we use a multivariate copula approach taking the state-varying dependence of business segments explicitly into account. A high level of a firm's diversification determined by a low dependence between the firm's business segments leads to a lower probability of firm default which results in a higher firm value through reduced bankruptcy costs. We demonstrate this effect by comparing the values of three U.S. firms when modeling independence, dependence with copulas, and perfect dependence between businesses.

JEL-Classification: G11, G17, G33, L25

Keywords: diversification, firm valuation, dependence modeling, multi-business firm, bankruptcy costs, default probability, copulas, Monte Carlo simulation, discounted cash flow model

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1 Introduction

In this article we present a multi-business firm valuation approach to examine how the level of diversification affects the firm value due to risk reduction. A high level of corporate diversification determined by a low dependence between a firm's business segments reduces the volatility of the entire firm's cash flow. It leads to a lower probability of firm default which results in a higher firm value through reduced bankruptcy costs. Since the co-movement of business segments is not constant across the business cycle, we apply a multivariate copula approach to model state-varying diversification.

The effect of diversification on firm value has been widely empirically examined in literature.[1] We extend the valuation literature and show how to relate the risk reduction effect of diversification directly to the valuation framework. Our model reveals several advantages. Analysts can determine the benefits that a diversified firm has through a decrease of default risk. The model reflects the diversification effect across the states of the economy by incorporating the existing asymmetric co-movements between businesses. For example, in a firm's business portfolio allocation, if all businesses plunge down jointly as the economy falls, the value of diversification may be overstated by those not taking the increase in downside co-movements into account. Through the explicit modeling of the different characteristics for each business segment, the model affords a more accurate estimate of the entire firm's value.[2] Moreover, the model provides the default probability for rating decisions.

Diversified firms have significantly lower cash flow volatilities. Dimitrov and Tice (2006) state that the greater cash flow volatility of less-diversified firms increases the costs of external financing and drives these firms to become credit-rationed during recessions. Especially during recessions and after exogenous industry shocks, when the cost of external financing increases

[1] See e.g. Lang and Stulz (1994), Berger and Ofek (1995), Servaes (1996), Lamont and Polk (2001), Campa and Kedia (2002), Graham et al. (2002), Villalonga (2004), Santalo and Becerra (2008), and Kuppuswamy and Villalonga (2010). There is no clear consensus regarding whether there is a discount or a premium on firm value.

[2] For discussion see Koller et al. (2005) and Damodaran (2009a).

and the business segments of a diversified firm would be financially constrained as stand-alone firms, diversification becomes more efficient. Yan (2006) and Yan et al. (2010) explain that under these circumstances diversified firms profit from their ability to substitute costly external capital with less costly internal capital. Diversified firms have better access to capital markets at lower costs when securing their financial needs. The lower costs result from risk reduction of lenders achieved through diversification, as described by Leland (2007). The lower cash flow volatility reduces downside risk and consequently leads to less defaults. Ammann and Verhofen (2006) show in a simulation study that the diversification influence on firm value depends on the correlation between business segments. More precisely, Lewellen (1971) postulates that as long as the cash flows of the various business segments are not perfectly positively correlated, the probability of default declines with the level of diversification.

As mentioned by Inderst and Müller (2003), the effect of shocks on business lending can be damped by internal capital markets of diversified firms which stabilizes the operating business. If a failing segment is short of liquidity, other business segments often provide cross-subsidies. Therefore, they can also get into trouble with liquidity. As also mentioned by Scharfstein and Stein (2000), weaker segments get subsidized by stronger ones through internal capital allocation. Meyer et al. (1992) argue that an unprofitable business segment can have negative value when it is part of a corporation that provides cross-subsidies, whereas a stand-alone firm would declare bankruptcy in the same scenario. Moreover, Lamont (1997) mention, when a business segment of a diversified firm is adversely affected by an exogenous shock, each of the other segments will cut investments by the same amount, regardless of whether they have comparatively better or worse opportunities than other segments. Billett and Mauer (2003) and Kuppuswamy and Villalonga (2010) examine the internal capital allocation in times of financing constraints and find that diversified firms have the ability to fund valuable projects of segments that would face binding financial constraints as stand-alone firms. Through these cross-subsidies in times of crisis the segment-specific cash flows are smoothed and the co-movement between the business segments increases resulting in a higher dependence in crisis. Erdorf and Heinrichs (2010) provide empirical evidence that the co-movement between industry revenues differs in the business cycle. In times of crisis, macroeconomic shocks pertain to almost all businesses and their revenues fall down together. The co-movement between revenues increases in crisis.

The crucial point in covering the risk-reduction effect of diversification explicitly is how to model the appropriate dependence between the business segments. Not only is the magnitude of dependence important, but also its variation in different states of environment and during the business cycle. Buraschi et al. (2010) model stochastic correlation. Their approach regards time-variant correlation but does not model structural changes of dependence. Our valuation model takes systematic differences in dependence between businesses into account by implementing a multivariate copula approach. Using the Clayton copula we are able to model higher dependence in crisis through asymmetry and tail dependence.

We focus on firms that are diversified across multiple lines of business.[3] Different businesses have different growth and risk characteristics. The growth and discount rates can vary widely across businesses. We therefore model each segment separately by applying an extended two-stage discounted cash flow model. We determine the revenues as the main value driver and follow Schwartz and Moon (2000) and (2001) by assuming a stochastic process for them. When simulating the revenues, we include dependence in the stochastic term by using copulas. We model segment-specific growth rates, volatilities of revenues, and discount rates to cover individual segment behavior and risk characteristics. Further items are developed proportionally to the revenues. A firm goes bankrupt when the sum of all segment-specific free cash flows falls below a predetermined amount of additional financing available for the entire firm. Section 2 explains the valuation model in detail.

In Section 3, we demonstrate how the model can be applied for illustrative examples. We show

[3] In our argumentation we do not discuss geographical diversification, but it could be easily incorporated in the model. Multi-country firms are widely examined in Damodaran (2009a), e.g. in terms of market-specific risk, growth rates, tax rates, and currencies.

for three diversified U.S. firms with varying numbers of business segments how the firm value changes if we vary the assumptions of the diversification magnitude. For each firm we consider three scenarios: We first assume perfect dependence between all business segments to simulate the effect of a non-diversified firm, and secondly, we model independence to simulate a perfect diversification between the segments. Beside those two extreme scenarios we capture the state-varying diversification effect by applying the copula approach to model the true dependence between the business segments. The three scenarios demonstrate the risk diversification effect which results in a higher firm value through a lower default probability. Finally, section 4 concludes the article.

2 Valuation Approach

Our model considers segment-specific cash flows and capital costs reflecting the differences in risk and growth characteristics across the different businesses that a firm may operate in. It adopts the diversification effect of multi-business firms. First, we describe the discounted cash flow model and second, we implement a bankruptcy barrier in the model to allow defaults. Third, we define the segment-specific free cash flow estimation and determine the stochastic process of the revenues. Fourth, we describe the multivariate copula approach and afterwards explain how to model segment-specific discount rates.

2.1 Discounted Cash Flow Model

For the valuation of multi-business firms we apply an extended discounted cash flow (DCF) approach.[4] The market value of an entire firm is determined by the discounted value of all expected future free cash flows aggregated over all segments. The free cash flow is defined as the cash flow available for distribution to all debt and equity security holders of the entire firm. In our approach we extend the typical WACC-model. We therefore discount the segment-specific

[4] When valuing multi-business firms, DCF models have several advantages over multiple approaches, see Damodaran (2009a).

free cash flows with a segment-specific weighted average cost of capital r_{WACC_l} at each time t and aggregate them to an enterprise value over all segments l (see equation (1)). The expected consolidated enterprise value is then determined by the arithmetic mean of all simulated paths i of the Monte Carlo simulation. Multi-business firms often have complex holding structures with minority holdings in subsidiaries and majority holdings in others. In the case of a majority holding, the balance sheets of the two firms are consolidated and the proportion of the firm that is owned by other investors is balanced as a minority interest.[5] To calculate the expected equity value of the consolidated firm V_0 conditioned on the information available at valuation date 0, we have to add market values of cash and marketable securities $CASH_0$ as well as minority holdings MH_0 and subtract the market values of minority interests MI_0 and interest bearing liabilities $Debt_0$ from the enterprise market value (see equation (2)):

$$V_{0,i}^D = \sum_{l=1}^{L} \sum_{t=1}^{\infty} \frac{FCF_{l,t}}{(1 + r_{WACC_l})^t} \tag{1}$$

$$V_0 = \mathbb{E}[V \mid \mathcal{F}_0] \approx \frac{1}{N} \sum_{i=1}^{N} V_{0,i}^D + CASH_0 + MH_0 - MI_0 - DEBT_0 \tag{2}$$

with paths index $i = 1, \ldots, N$; segment index $l = 1, \ldots, L$ and time index $t = 1, \ldots, \infty$.

Where:

$V_{0,i}^D$	= enterprise market value of corporation's simulated path i at valuation date 0
V_0	= equity market value of the corporation at valuation date 0
$FCF_{l,t}$	= free cash flow for segment l at time t
r_{WACC_l}	= time invariant weighted average cost of capital for segment l
$CASH_0$	= market value of cash and marketable securities at valuation date 0
MH_0	= market value of minority holdings at valuation date 0

[5] See Damodaran (2009a) for an explanation about the valuation of complex holding structures, such as minority holdings and minority interests.

MI_0 = market value of minority interests at valuation date 0

$DEBT_0$ = market value of debt as the sum of interest bearing liabilities at valuation date 0

The DCF approach requires all expected future free cash flows to be forecasted ad infinitum. To avoid this problem, we extend the above DCF model to the following two stage DCF version by separating the expected free cash flows into two periods. The two-stage DCF model assumes that in a first stage forecasts of the free cash flows are made on a period-by-period basis. In the first stage, the explicit forecast period, we calculate the present value by discounting segment-specific free cash flows for a limited number of periods ($t = 1, \dots, T$). After this stage, individual forecasts are typically difficult to predict and the firm is usually assumed to achieve a period of steady but adequate growth. When a firm operates in multiple businesses, some of these segments might have high growth rates, whereas others are already in stable growth. The explicit forecast period should be long enough to ensure that all business segments reach stable growth before entering the second stage. In this terminal period, we make the assumption that all segments have reached steady state and use a perpetuity expression.

$$V_{0,i}^D = \sum_{l=1}^{L} \left(\sum_{t=1}^{T} \frac{FCF_{l,t}}{(1 + r_{WACC_l})^t} + \frac{FCF_{l,T+1}}{(1 + r_{WACC_l})^T \cdot (r_{WACC_l} - g)} \right) \tag{3}$$

where g is an expected constant growth rate.

It follows for V_0:

$$\begin{aligned} V_0 = \frac{1}{N} \sum_{i=1}^{N} \left(\sum_{l=1}^{L} \left(\sum_{t=1}^{T} \frac{FCF_{l,t}}{(1 + r_{WACC_l})^t} + \frac{FCF_{l,T+1}}{(1 + r_{WACC_l})^T \cdot (r_{WACC_l} - g)} \right) \right) \\ + CASH_0 + MH_0 - MI_0 - DEBT_0 \end{aligned} \tag{4}$$

In the steady state all business segments grow at this expected constant growth rate g and reinvest a proportion of their profits into the businesses each year. The expected constant growth rate g is specified exogenously.[6] The steady state requires that all items in financial statements are modeled in a consistent manner. Particularly, in the terminal value period each item grows towards infinity at the same growth rate g.[7] The terminal value captures the present value of the expected free cash flows beyond the explicit forecast period, which is computed on an extrapolation of the last detailed free cash flow at time T based on growing perpetuities and steady state assumptions.

2.2 Probability of Default

As Damodaran (2009b) points out, when the possibility of default is neglected, DCF valuations overstate the firm value significantly. The diversification level of the firm's portfolio of business segments impacts the cost of bankruptcy since diversification reduces the downside risk ceteris paribus. A low level of a firm's diversification is determined by a high dependence between the firm's business segments. A higher dependence results in a higher default probability and leads to a lower firm value. On the other hand, firms can benefit from a high level of diversification due to the reduction of downside risk. For example, if a business segment performs poorly and would default as a stand-alone firm, it can be rescued by other segments of a diversified firm which generate positive cash flows.

We assume that all free cash flows are completely distributed to debt and equity holders each period. The entire firm is assumed to default when its total free cash flow, which is determined as the sum of segment-specific free cash flows, falls below a predetermined amount of additional financing. We define default as the first time when $\sum_{l=1}^{L} FCF_{l,t}$ hits the negative amount of additional financing.

$$\sum_{l=1}^{L} FCF_{l,t} < -AF \tag{5}$$

$$AF = \text{Amount of potential additional financing}$$

[6] Note, g is often linked to the expected average inflation rate or to the growth in the nominal gross domestic product (GDP).

[7] Lundholm and O'Keefe (2001) examine in more detail financial statements in steady state.

If the firm defaults in a path of the Monte Carlo simulation, its value for this path is set to zero assuming that the liquidation proceeds and the firm value until default equals the bankruptcy costs. This modeling of default has several advantages. First, a diversified firm only defaults when its sum of all free cash flows falls below a predetermined amount, considering the possibility that other business segments can provide cross-subsidies when a failing segment is short of liquidity. Second, the model can consider additional financing. Finally, our approach measures the cumulative joint probability of default as the proportion of simulations where the firm goes bankrupt. This could be a helpful source for rating decisions of corporate bonds.[8]

2.3 Free Cash Flow Estimation

The segment-specific free cash flow ($FCF_{l,t}$) is computed by the following formula:[9]

$$FCF_{l,t} = (R_{l,t} - C_{l,t} - DE_{l,t})(1 - \tau) - (CE_{l,t} - DE_{l,t}) - \Delta WC_{l,t} \quad (6)$$

where:

$R_{l,t}$ = revenues for segment l at time t

$C_{l,t}$ = sum of cost of goods sold (COGS) and operating expenses (OEXP) for segment l at time t

$DE_{l,t}$ = depreciation and amortization for segment l at time t

τ = corporate tax rate (assumed to be constant over segments and time)

$CE_{l,t}$ = capital expenditures in net property, plant and equipment (PPE) for segment l at time t

$\Delta WC_{l,t}$ = change in non-cash operating working capital for segment l at time t ($\Delta WC_{l,t} = WC_{l,t} - WC_{l,t-1}$)

[8] An application of cumulative probability of default for bond rating is shown in Altman and Pasternack (2006).

[9] See Damodaran (2006). Note, we omit changes in deferred taxes.

The free cash flow is equal to after-tax operating profit, less investments in net property, plant and equipment ($CE_{l,t} - DE_{l,t}$) and changes in non-cash operating working capital ($\Delta WC_{l,t}$). It does not incorporate any financing-related cash flows such as interest expenses or dividends.

Since the revenues are one of the key value drivers of the firm value they are the central variable and we model them explicitly. We follow Schwartz and Moon (2000) and (2001) and assume in a continuous time approach that the dynamics of the revenues of business segment l are given by a stochastic differential equation. We therefore specify them by the following geometric Brownian motion:[10]

$$\frac{dR_{l,t}}{R_{l,t}} = \mu_{l,t}\ dt + \sigma_l\ dZ_{l,t} \tag{7}$$

The drift $\mu_{l,t}$ is the expected rate of growth in segment-specific revenues, and σ_l is the volatility of segment-specific revenues. Our approach models revenues for each segment separately taking into account that growth rates and volatilities vary widely across businesses. We assume that the expected rate of growth $\mu_{l,t}$ is described by a deterministic mean reversion with a long-term average drift $\overline{\mu}$.[11]

$$d\mu_{l,t} = \kappa \cdot [\overline{\mu} - \mu_{l,t}]dt \tag{8}$$

The initial growth rate $\mu_{l,t}$ specified by a business segment l converges to the sustainable growth rate $\overline{\mu}$. We consider that high growth rates in segment-specific revenues would lead to competitors who imitate or copy business ideas and products. High growth rates can not survive to the long term horizon. The mean reversion coefficient κ describes the adaption speed at which the segment's growth is expected to converge deterministically to the long term average. According to Schwartz and Moon (2000) and (2001), we interpret $\frac{\ln(2)}{\kappa}$ as the half-life of the deviations in

[10] Furthermore, the revenues can be modeled in a more advanced approach by a fractional Brownian motion which is based on Mandelbrot and Ness (1968).

[11] Schwartz and Moon (2000) and (2001) apply a stochastic differential equation for modeling the expected growth rate in revenues. Since we do not focus on startups in our model, the above approach simplifies the model.

which the growth rate is expected to be halved. The volatility σ_l in equation (7) is specified for each segment l separately and it is assumed to be constant over all time points t.

While in Schwartz and Moon (2000) and (2001) $Z_{l,t}$ is a Wiener process, a more general approach could be implemented. $Z_{l,t}$ can be chosen as any stochastic process according to the specific properties of each segment. To overcome the shortcomings of the Brownian motion, especially the normally distributed increments, other Lévy processes can be applied. The flexibility of the copula approach allows us to be free in the choice of the marginal distributions of the segments' revenue dynamics. The multivariate distribution of the resulting multivariate process can then be modeled by arbitrary individual distributions combined with the appropriate copula which introduces dependence between the $Z_{l,t}$.

In addition to the revenues, the other items of the free cash flow formula (6) are linked to these stochastically modeled revenues. We apply the textbook approach of financial planning, the percentage of sales method (POS). In compliance with Koller et al. (2005), the growth in revenues determines the development of most items in the financial statement directly. As an illustration, one can consider a firm which is operating efficiently and at exhausted capacity. If it wants to increase its revenues by a certain percentage, it basically has to enhance all items on the asset side of the balance sheet by the same percentage. On the liability side, a part of this amount could be financed by increasing liability items, e.g. accounts payable linked to the revenues which are considered in the change of operating working capital. The remaining funds have to be raised by enhancing equity and debt, e.g. through increases in share capital and long term debt. Principally, a firm could finance growth by only using new debt. It results in changes of the firm's leverage ratio and effects the weighted average cost of capital. In our model we assume a policy of constant leverage ratio which implies that the firm increases both equity and debt proportionally.

As already mentioned, further items will be tied directly to revenues. More precisely, the items

$CE_{l,t}$ and $WC_{l,t}$ are modeled as a certain ratio δ_l^{CE} and δ_l^{WC} of the revenues for segment l at time t:[12]

$$CE_{l,t} = \delta_l^{CE} \cdot R_{l,t} \tag{9}$$

$$WC_{l,t} = \delta_l^{WC} \cdot R_{l,t} \tag{10}$$

Changes in non-cash operating working capital $\Delta WC_{l,t}$ are defined as the differences between non-cash operating working capital at the end of the period t and non-cash operating working capital at the end of the previous period $t-1$. It contains the net investments in accounts receivable, inventory, accounts payable and accruals that are requested to support growth in revenues.

Depreciation and amortization $DE_{l,t}$ are assumed to be a fixed percentage δ_l^{DE} of net property, plant and equipment $PPE_{l,t}$ for segment l at time t:

$$DE_{l,t} = \delta_l^{DE} \cdot PPE_{l,t} \tag{11}$$

The net property, plant and equipment increases with capital expenditures and decreases with depreciation. We therefore calculate changes in net property, plant and equipment $\Delta PPE_{l,t}$ by the difference between capital expenditures and depreciation and amortization:

$$\Delta PPE_{l,t} = PPE_{l,t} - PPE_{l,t-1} = CE_{l,t} - DE_{l,t} \tag{12}$$

Using equation (11) in equation (12), it follows:

$$PPE_{l,t} = \frac{PPE_{l,t-1} + CE_{l,t}}{(1 + \delta_l^{DE})} \tag{13}$$

[12]If long-term estimates for capital expenditures are available, they can of course be incorporated directly into the valuation model.

Using net property, plant and equipment $PPE_{l,t}$ in expression (11) leads to the amount of depreciation and amortization $DE_{l,t}$.

As the next item of equation (6) we consider the cost component. To separate between fixed and variable costs, the costs $C_{l,t}$ are determined by linear regression. Related to Schwartz and Moon (2000), we subdivide the total costs $C_{l,t}$ into two components for segment l at time t. The first part is the cost of goods sold $COGS_{l,t}$ which is modeled by a fixed proportion to the revenues. The second term are the other expenses $OEXP_{l,t}$ which consist of fixed costs F_l and variable costs.

$$\begin{aligned} C_{l,t} &= COGS_{l,t} + OEXP_{l,t} \\ &= \beta_{l,1} \cdot R_{l,t} + (F_l + \beta_{l,2} \cdot R_{l,t}) \\ &= F_l + \tilde{\beta}_l \cdot R_{l,t} \end{aligned} \tag{14}$$

where:

$\beta_{l,1}$	= COGS as a percentage of revenues for segment l
$\beta_{l,2}$	= variable costs of other expenses as a percentage of revenues for segment l
$\tilde{\beta}_l$	= sum of variable costs ($\beta_{l,1} + \beta_{l,2}$) as a percentage of revenues for segment l
F_l	= fixed costs for segment l

The cost components are specified by this linear regression model with fixed costs F_l as the intercept and $\tilde{\beta}_l$ as the slope coefficient. The fixed costs are modeled segment-specific and assumed to be time-constant. The variable costs are a proportion of the time-varying segment-specific revenues.

Multi-business firms have general expenses on the corporate level, e.g. for reasons of corporate control and shared services to avoid multiple divisions of accounting and human resources.

These expenses can be allocated across business segments, e.g. according to their proportion of revenues, or can be retained at the corporate level. To model a separate corporate segment, revenues generated at the corporate level are required in our model.

To determine the ratios segment-specific data are required. According to US-GAAP and IAS/IFRS accounting, publicly listed firms are obligated to report certain data about their segments. SFAS No.131 and IFRS 8 require multi-business firms to disclose revenues, operating profit, assets, capital expenditures and depreciation and amortization, among others, for their business segments.[13] If nevertheless not reported, the segment-specific data can be estimated by deriving a segment ratio based on the proportion of segment-specific revenues to a corporation's total revenues. As an alternative, industry averages of stand-alone firms can be applied to estimate segment-specific data, e.g. segment-specific non-cash operating working capital is not provided by segment reporting. To obtain it, we first determine the segment weight w_l as the proportion of segment-specific revenues to corporation's total revenues. Then, the proportion of initial segment-specific non-cash operating working capital $WC_{l,0}$ is estimated by multiplying total non-cash operating working capital WC_0 with the segment weight w_l:

$$WC_{l,0} = w_l \cdot WC_0 \tag{15}$$

As discussed by Koller et al. (2005), intercompany revenues do not affect the free cash flow and the value of the corporation or the individual business segments. One segment reports the item as costs and the other as revenues and the entire firm overall nets it out. We assume that intercompany revenues are sold at cost and therefore have no impact on the operating profit. As a consequence, we eliminate internal revenues by only using external revenues. Business segments with shortages of liquidity can receive cash from other business segments leading to intercompany payables. These internal receivables and payables should not be considered as part of the operating working capital. We avoid this by applying the consolidated non-cash

[13] Analysts can employ segment-specific data from `COMPUSTAT` or Business Information Tracking Series (BITS).

operating working capital of the entire firm. Finally, we assume a constant corporate tax rate τ for all segments l and all time points t.[14]

2.4 Introducing Dependence with Copulas

To model different diversification levels, we implement dependence between the revenues of business segments that a firm operates in. Since the diversification effect depends heavily on the economic state and varies across the business cycle, the implementation of copulas as a dependence structure enables us to incorporate this effect into valuation. The dependence is introduced between the Wiener processes $Z_{l,t}$ in equation (7). To obtain the joint distribution of $(Z_1, \ldots, Z_L)$ we apply a multivariate copula approach.

Copula functions are L-dimensional distribution functions $C : [0,1]^L \to [0,1]$ with standard uniform marginal distributions. They present a flexible instrument for modeling any dependence structure of random variables. Given the marginal distributions, this statistical tool enables us to uniquely specify the joint distribution function by the choice of any copula. The unique connection between marginal distributions and joint distributions via copulas is described by Sklar's Theorem which remarks the cradle of copula theory. Sklar (1959) states that any distribution function $F(z_1, \ldots, z_L)$ of a continuous random variable $(Z_1, \ldots, Z_L)$ can be represented in the form

$$F(z_1, \ldots, z_L) = C(F_1(z_1), \ldots, F_L(z_L)), \tag{16}$$

where $F_1, \ldots, F_L$ are the marginal distributions of $Z_1, \ldots, Z_L$ and $C : [0,1]^L \to [0,1]$ is an L-variate copula with $z_1, \ldots, z_L \in \bar{\mathbb{R}} = [-\infty, \infty]$. Conversely, if C is a copula and $F_1, \ldots, F_L$ are univariate distribution functions, then the function F defined in (16) is a joint distribution function with margins $F_1, \ldots, F_L$.

[14]Note, if a multi-business firm operates in different countries, it might be necessary to consider country-specific tax rates. Analysts should regard whether the firm can transfer its earnings into countries with lower tax rates and whether interest expenses can be linked to that country which maximizes tax benefits.

This approach allows us to specify the marginal distributions of the segment-specific revenues independently and afterwards apply an adequate copula to achieve dependence between them. This yields a multivariate distribution of business segments. One of the most famous and often applied copulas is the class of elliptical copulas, especially the Gaussian copula. This copula has the drawback of radial symmetry which leads to the same dependence in times of crisis and boom. The dependence structure would be linear and following our assumptions would lead to an underestimation of bankruptcy costs in a valuation model. To avoid this, we use a more flexible copula which models asymmetry and stronger tail dependence, the so-called Clayton copula. It belongs to one of the most important classes of copulas, the Archimedian copulas. These copulas have several advantages compared to elliptical copulas, such as the Gaussian. For example, all commonly encountered Archimedian copulas have closed form expressions. In general, Archimedian copulas are constructed with help of a generator function as follows:

$$C(u_1, \dots, u_L) = \varphi^{-1}\left(\sum_{i=1}^{L} \varphi(u_i)\right), \tag{17}$$

where $\varphi : [0,1] \to [0,\infty]$ is a continuous, strictly decreasing and convex function such that $\varphi(1) = 0$ and $\varphi(0) = \infty$. The function φ is called the generator of the copula. The generator of the Clayton copula, which is also known as the Cook-Johnson copula, is

$$\varphi_\theta(t) = \frac{1}{\theta}(t^{-\theta} - 1), \qquad \theta > 0. \tag{18}$$

Thus, according to equation (17), it follows for the Clayton copula :

$$C_\theta^{CJ}(u_1, \dots, u_L) = \left[\left(\sum_{i=1}^{L} u_i^{-\theta}\right) - L + 1\right]^{-\frac{1}{\theta}} \tag{19}$$

The dependence parameter θ can be estimated with a maximum-likelihood approach in practice. As shown in figure 1, the Clayton copula exhibits an asymmetric dependence structure with lower tail dependence but no upper tail dependence.[15] This allows us to model different dependence for times of crisis and boom. This is important, especially during a crisis, as the dependence between segments is much higher compared to the common phases of the business cycle.[16] A more detailed discussion about Archimedian copulas can be found in Joe (1997), Embrechts et al. (2003) and Nelsen (2006), for example.

[Please insert figure 1 here]

Most copulas, such as the Clayton copula, are only appropriate for the bivariate case as they have very restrictive assumptions, such as equal rank correlations between the random variables. In the bivariate case we could only focus on two business segments, but multi-business firms are often comprised of more than two segments. To circumvent this challenge, we apply a new technique of pair-copula construction.

The pair-copula construction was introduced by Aas et al. (2009) as a framework to overcome the difficult task of constructing multivariate distributions with copulas. They use a hierarchical approach of cascading pair copulas to obtain a multivariate dependence structure by applying only bivariate copulas. It is reached by decomposing an L-variate copula into a product of $\frac{L(L-1)}{2}$ bivariate copulas. There are plenty of possible ways of decomposing a density function. As a graphical intuitive way, Bedford and Cooke (2001) introduce the regular vines. We apply the so-called D-vines as a class of regular vines. A complete introduction of vine copulas is beyond the scope of this article but it can be found in Bedford and Cooke (2001) and Aas et al. (2009). The decomposition can be derived by starting with Sklar's Theorem. First we take partial derivatives with respect to both arguments in $F(z_1, z_2) = C(F_1(z_1), F_2(z_2))$ and obtain

$$f(z_1, z_2) = c_{12}(F_1(z_1), F_2(z_2)) \cdot f_1(z_1) \cdot f_2(z_2), \tag{20}$$

[15] Moreover, the fact that the Clayton copula can only model positive dependence is a drawback only at first glance. In financial applications, and also when modeling revenues, positive dependence is much more present than negative.

[16] For empirical evidence between industries see Erdorf and Heinrichs (2010).

which is equivalent to

$$f_{2|1}(z_2|z_1) = \frac{f(z_1, z_2)}{f_1(z_1)} = c_{12}(F_1(z_1), F_2(z_2)) \cdot f_2(z_2), \tag{21}$$

and leads to the general formula

$$f_{j|i}(z_j|z_i) = \frac{f(z_i, z_j)}{f_i(z_i)} = c_{ij}(F_i(z_i), F_j(z_j)) \cdot f_j(z_j). \tag{22}$$

As an example, we consider the 4-dimensional case assuming a multi-business firm that operates four businesses. The density function f can be decomposed as follows:

$$f(z_1, z_2, z_3, z_4) = f_1(z_1) \cdot \underbrace{f_{2|1}(z_2|z_1)}_{*} \cdot \underbrace{f_{3|1,2}(z_3|z_1, z_2)}_{(**)} \cdot \underbrace{f_{4|1,2,3}(z_4|z_1, z_2, z_3)}_{(***)}. \tag{23}$$

Using equation (22) repeatedly, we obtain the following decomposition of f as an expression which only consists of bivariate copula densities and univariate densities $f_1, \ldots, f_4$:

$$\begin{aligned}
f(z_1,z_2, z_3, z_4) = f_1(z_1)\cdot & & (24)\\
& c_{12}(F_1(z_1), F_2(z_2)) \cdot f_2(z_2)\cdot & (*)\\
& c_{13|2}(F_{1|2}(z_1|z_2), F_{3|2}(z_3|z_2)) \cdot c_{23}(F_2(z_2), F_3(z_3)) \cdot f_3(z_3)\cdot & (**)\\
& c_{14|23}(F_{1|23}(z_1|z_2, z_3), F_{4|23}(z_4|z_2, z_3)) \cdot c_{24|3}(F_{2|3}(z_2|z_3), F_{4|3}(z_4|z_3))\cdot & (***)\\
& \qquad c_{34}(F_3(z_3), F_4(z_4)) \cdot f_4(z_4) &
\end{aligned}$$

As illustrated in figure 2, a D-vine copula is a decomposition of a multivariate copula to a product of bivariate copulas. The figure shows a nested set of three trees. The first tree consists of three bivariate copulas, the second tree consists of two copulas and the third tree consists of one.

[Please insert figure 2 here]

After having evaluated segment-specific free cash flows and introduced dependence between them, we take a closer look into the discount factors.

2.5 Cost of Capital

We consider uncertainty with a risk premium in the discount rates. As the appropriate discount rate we apply segment-specific weighted average cost of capital r_{WACC_l} to cover the different risk characteristics of different businesses. It consists of the weighted average of the cost of equity capital and the cost of non-equity capital. The r_{WACC_l} is defined as:

$$r_{WACC_l} = r_{E_l} \cdot w_E + r_D \cdot (1 - \tau) \cdot w_D \tag{25}$$

where:

r_{E_l} = segment-specific cost of equity for segment l

r_D = corporate cost of debt as a weighted average of the cost of interest bearing liabilities

w_E = fixed proportion of equity in corporate capital structure

w_D = fixed proportion of debt in corporate capital structure

τ = corporate tax rate (assumed to be constant over segments and time)

Since operating risk differs across segments, we consider equity risk premiums for every industry that a firm operates in. We apply segment-specific cost of equity r_{E_l} which comprises the industry risk premium.[17] We presume that business segments which are combined in a corporation have a common optimal financial capital structure.[18] According to Koller et al. (2005), multi-business firms typically manage debt centrally for all business segments. As Damodaran (2009a) points

[17] Fama and French (1997) argue that industry cost of equity is more accurate than firm-specific cost of equity. If a multi-business firm operates in different countries, the discount rates should be higher in riskier markets, e.g. in emerging markets, than in developed markets.

[18] For discussion see Leland (2007).

out, debt used by the firm is not broken down by business segments and the market value of equity is accessible only for the entire firm and not available for individual segments. We assume that a firm uses the same mix of debt and equity across all business segments and therefore all segments have the same capital structure.[19] Furthermore, the cost of debt r_D is modeled constant across all segments.

We assume r_{WACC_l} to be invariant over all time points. Since capital weights have to be derived from market values, the typical WACC-model encloses circularity problems.[20] To avoid this, we simplify the model and assume a time constant expected financial leverage ratio. A policy of constant leverage ratio implies that the firm has to increase both equity and debt proportionally if new capital is required. In particular, the capital structure, and accordingly the weights w_D and w_E, are time invariant. To circumvent the circularity problems, we further define time invariant cost of debt r_D and cost of equity r_{E_l}. We assume that a business segment continues in its existing business and does not change its operating risk. By assuming a policy of constant financial leverage ratio, the financial risk does also not vary over time. Therefore, we assume r_{WACC_l} to be invariant over all states of the economy. According to the state price theory, we assume given $r_{WACC_{l,j}}$ for all possible states j of the economy. To obtain state independent r_{WACC_l} for each segment l, we use the sum of all probability weighted state-given $r_{WACC_{l,j}}$. This simplification is unproblematic because the assumption of state independent r_{WACC_l} leads to an underestimation of the actual effect of bankruptcy costs. Any assumption of higher costs in times of crisis results in higher bankruptcy costs through more frequent defaults.[21]

[19] As an alternative modeling, the use of an industry average capital structure of stand-alone firms in the same business could be applied for a segment-specific capital structure. However, a disadvantage is that stand-alone firms have a smaller debt capacity than multi-business firms, as described by Lewellen (1971), and that the debt across segments would not sum up to corporate debt.

[20] Courteau et al. (2001) derive a feasible implicit WACC-model where circularity problems are avoided.

[21] For example, state-varying cost of debt could be considered by implementing financial rating stages in the model. If the cash flow of the entire firm falls below a stage, the cost of debt increases, and if the cash flow reaches an upper stage, the cost of debt decreases.

3 Illustrative Examples

To illustrate the diversification effect on firm value, we implement the methodology for valuing a corporation by applying it to three public listed U.S. firms with varying numbers of business segments. For each firm we consider three scenarios: First, we assume perfect dependence between all business segments to simulate the effect of a non-diversified firm, and second, we model independence to simulate a perfect diversification between the segments. Beside those two extreme scenarios we capture the state-varying diversification effect by applying the copula approach to model dependence between the business segments. In this section we introduce the three examined firms and explain how we estimate the valuation's parameters. Afterwards we describe the simulation design and analyze the findings.

3.1 Firms and Business Segments

We value Servotronics Inc. (operating in two unrelated businesses)[22], Sensient Technologies Corp. (operating in three related businesses) and Flowserve Corp. (operating in four unrelated businesses) at the end of 2009.[23] The business segments are obtained from the `COMPUSTAT` segment database. Table 1 reports the three firms with their associated business segments and their classification by SIC codes and Fama-French industry groups.[24]

[Please insert table 1 here]

Analysts should notice that the use of segment data can be challenging since the definition of business segments is flexible. Business segments are self-reported and the aggregation of activities into single business segments therefore differs across firms.[25] Firms sometimes change the

[22]Segments are classified as unrelated if they do not share a common 2-digit SIC code, and are labeled related otherwise.

[23]Note that for reasons of clarity and comprehensibility, we only consider up to four business segments per firm. An extension to more segments can be easily implemented.

[24]The business segments are classified by their four-digit SIC codes into one of the 48 Fama-French industry groups. For the transformation of SIC codes into these industries see Appendix of Fama and French (1997).

[25]Davis and Duhaime (1992) show that in 5 to 10 per cent of their cases, businesses which are neither related nor vertically integrated, are combined into a single segment by the firms.

classification of their reporting segments, although there is no real underlying change in their operations. Segment increases documented by COMPUSTAT do not necessarily represent actual diversifying events but they can be the result of reporting changes (e.g. Denis et al. (1997) and Graham et al. (2002)). Hayes and Lundholm (1996) argue that segment reporting is often distorted by strategical management decisions. As Villalonga (2004) dicusses, a strategic accounting explanation suggests that managers report segment data in ways that make them appear to be worse performers than they actually are, in order to avoid disclosing valuable information to competitors. Hence, when employing COMPUSTAT segment data, there is considerable risk that diversification is not measured correctly in the first place, which could in turn introduce bias in the assessment of diversification's effects on firm value. Therefore, analysts should take care of segment reporting and analyze each firm separatly to cover all segment-specific characteristics.[26]

3.2 Parameter Estimation

The valuation model, as described in section 2, requires more than 20 parameter estimations for its implementation. Some of these initial items are easily observable from the last segment's financial statement. Others can be based on analyst forecasts or estimated on historical quarterly or yearly data available for business segments. However, the determination of some parameters requires the use of economic considerations. Table 2 summarizes the parameters of the model and gives an impression about their estimation. Afterwards, we describe in more detail how we estimate several input parameters in our examples.

[Please insert table 2 here]

The historical segment-specific data are obtained from the COMPUSTAT segment database. The segment data are provided yearly and conform to SFAS No.131. For the initial segment-specific

[26]When modeling multi-business firms we will be faced with information gaps and more complexity than in the case of modeling stand-alone firms. Individual segment characteristics, the firm's complex holding structures and the mentioned problems of COMPUSTAT data make it unworkable to develop an algorithm for a large sample study.

revenues we use the `COMPUSTAT`-item *Net Sales*. It only includes sales to customers and does not consider inter-segment revenues. Therefore, we do not need to correct it for revenues generated from sales to other business segments within a firm. The mean and standard deviation of growth rates in revenues are calculated by using the growth in revenues from the last 10 years. To regress costs on revenues, we compute historical segment-specific costs for the last 10 years by subtracting the segment items *Depreciation and Amortization* and *Operating Profit* from *Net Sales*. Besides these items, `COMPUSTAT` also offers segment-specific *Capital Expenditures*.

Additionally, aggregated data for each entire firm are obtained from the `COMPUSTAT` annual database. The items *Revenue (REVT)*, *Working Capital (WCAP)*, *Cash and Marketable Securities (CHE)* and *Net Property, Plant and Equipment (PPENT)* are used to estimate further parameters as described in table 2. The total non-cash working capital of an entire firm is calculated as the difference between *Working Capital (WCAP)* and *Cash and Marketable Securities (CHE)*. We compute the weight w_l and the ratios δ_l^{CE}, δ_l^{WC} and δ_l^{DE} by using arithmetic means based on historical data from the last 10 years. The firms exhibit no minority holdings in other firms but Flowserve reports minority interests. We approximate the market value of minority interests by their book value of the item *Minority Interests (MIB)*.[27] The market value of debt for each entire firm is approximated by the book values of the items *Debt in Current Liabilities (DLC)* and *Long Term Debt (DLTT)*.

The cost of debt is measured as the ratio of reported interest on debt to the book value of debt of the previous period, while the item *Interest and Related Expense (XINT)* denotes the interest expenses. The corporation's market value of equity is determined by the item *Market Value (MKVALT)*. The segment-specific cost of equity is computed by the sum of the current five-year Treasury constant maturity rate[28] as a measure for the risk-free rate and the Fama-French industry risk premium (48 industry code). To obtain an industry risk premium for each segment,

[27] For an alternative modeling of minority interests see e.g. Damodaran (2009a).

[28] Historical data of T-bond rates are available on the website of The Federal Reserve Board, http://www.federalreserve.gov/datadownload/.

the business segments are classified into one of the 48 Fama-French industries. We calculate industry risk premiums from the three-factor Fama-French model using a regression over the last five years and coefficients average of the Fama-French factors over the last 50 years.[29] We compute the capital weights w_D and w_E by using arithmetic means based on historical market values of equity and book values of debt from the last 10 years.

The long term average growth rate is chosen to be equal to the expected constant growth rate g and is set to an expected average inflation rate of three percent.[30] To simplify, the half-life of the mean reversion process is assumed to be two years for all segments. The corporate tax rate is assumed to be constant and is set to 35 percent. Furthermore, to allow additional financing in the case of shortage in liquidity, we implement a corporate bankruptcy barrier which is approximated by five percent of the initial firm's total revenues. An overview of the parameter estimations used in the model to estimate the values of the three firms is presented in table 3.

[Please insert table 3 here]

We derive the input parameters of the D-vine copula structure from the growth rates of the segment-specific revenues. To derive these thetas, we transform the growth rates to uniform U(0,1) variables $u_{l,t}$ with the empirical cumulative distribution function. First, we assume that all bivariate copulas that the multivariate copula is decomposed to are Clayton copulas. In the bivariate case, the log-likelihood function is given by:

$$\mathcal{L}_{Clayton}(\theta; u_{1,t}; u_{2,t}) = \sum_{t=1}^{T} log\left((1+\theta)(u_{1,t} \cdot u_{2,t})^{-1-\theta}(u_{1,t}^{-\theta} + u_{2,t}^{-\theta} - 1)^{-2-\frac{1}{\theta}}\right) \tag{26}$$

[29]Detailed information about the Fama-French 48 industry returns and the Fama-French factors is available at the website of Kenneth R. French, http://mba.tuck.dartmouth.edu/pages/faculty/ken.french.

[30]Note that as an alternative modeling, analysts can assume that the segments first converge to the average industry growth in the explicit forecast period and afterwards grow with an expected inflation rate in the terminal period.

Second, we estimate the parameters of the D-vine structure by optimizing the copula vine log-likelihood as the sum of $\frac{L(L-1)}{2}$ bivariate likelihoods.[31] The copula parameters are assumed to be time invariant. Table 4 presents estimations of the thetas for the three firms.

[Please insert table 4 here]

3.3 Simulation Design

We apply Monte Carlo simulation to solve for the values of the three U.S. firms. To perform the simulations we use the discrete versions of equations (7) and (8):

$$R_{l,t+\Delta t} = R_{l,t} \cdot \exp\left\{[\mu_{l,t} - \frac{\sigma_l^2}{2}]\Delta t + \sigma_l \cdot \sqrt{\Delta t} \cdot Z_{l,t}\right\} \tag{27}$$

$$\mu_{l,t} = \overline{\mu} + \exp\left\{-\kappa \cdot t\right\}(\mu_{l,0} - \overline{\mu}) \tag{28}$$

We choose $Z_{l,t}$ for firm's business segments $l = 1, \ldots, L$ to be standard Wiener processes and implement different dependencies between them. For the two extreme scenarios we draw the $Z_{l,t}$ independently in the independent scenario and we apply equal $Z_{l,t}$ for all l in the scenario of perfect dependence. In the third scenario, we apply the copula approach using the algorithm for D-vine structures as described in Aas et al. (2009) to capture the true diversification effect.

The discrete time interval is chosen to be one year according to data availability as provided by the `COMPUSTAT` segment database. An advantage to using yearly data is that seasonal influences are smoothed. As mentioned by Ohlson and Zhang (1999), the valuation error is reduced by every increase in the explicit forecast horizon leading to a smaller effect of the terminal value on a firm's value estimates. We therefore choose 25 years to determine the explicit forecast period. After 25 years we assume that the business segments reach the terminal period with steady growth. To model a broad spectrum of possible growth paths for all segments, we perform

[31] The algorithm of the likelihood evaluation for a D-vine structure is described in Aas et al. (2009). We use the approach of Vogiatzoglou (2009).

1,000,000 simulations. If a simulated firm value of one path is negative, we set the value to zero. The estimated values of the firms are stable due to the number of simulations.

3.4 Simulation Results

The findings indicate the diversification effect on default risk and firm value. A high level of a firm's diversification is determined by a low dependence between the firm's business segments. We simulate three cases: scenario A with independence between the segments, scenario B with state-varying dependence modeled by the Clayton copula approach, and scenario C with perfect dependence. We assume that in the cases of independence and perfect dependence the diversification effect does not vary across the business cycle. The results of table 5 show that the higher the level of diversification is, the lower the default probabilities are. The higher level of diversification leads to a downside risk reduction because distressed business segments can be subsidized by others. These might not be distressed since they depend less on each other. The lower percentage of default results in higher firm values. This is caused by the fact that lower probabilities of default make it more likely that opportunities of future earnings can be achieved and therefore the present value of future cash flows is higher. The diversification effect across the scenarios is even underestimated in our simulation, as the capital costs should be lower in the scenario of independence and higher in the scenario C of perfect dependence. This would lead to even higher intrinsic equity values in scenario A and lower intrinsic equity values in scenario C.

[Please insert table 5 here]

Table 5 presents the simulation results of the three diversification scenarios for the three U.S. firms. The relative differences between the equity values are higher between scenario B and C than between scenario A and B. It indicates that the true diversification is closer to independence than to perfect dependence for the three firms. This finding is confirmed by the absolute differences between the scenarios of the 10-year and 25-year cumulative probabilities of default.

The intrinsic equity value of Servotronics simulated under the true dependence (scenario B) is \$34.49 million. The market capitalization at the end of 2009 was \$21.09 million, making Servotronics significantly under valued. In contrast, since the simulated equity value of Sensient Technologies is \$479.26 million and the market capitalization was \$1,282.70 million at the valuation date, Sensient Technologies is significantly over valued. Flowserve exhibits a simulated equity value of \$5,361.58 million and had a market capitalization of \$5,194.99 million at the end of 2009, making it slightly under valued. The results obtained from the valuation model depend on the assumptions we make about the parameters.

4 Conclusion

The intent of this article is to implement a new valuation model which adopts the diversification effect of multi-business firms. We show how different diversification extents can affect the firm value due to risk reduction. To consider business segment's particular growth and risk characteristics, we implement stochastic processes and simulate future cash flows for each segment separately. In illustrative examples, we show for three diversified U.S. firms how the firm value changes if we vary the assumptions of the diversification degree implemented by the dependence between the business segments. Three scenarios demonstrate the risk diversification effect which results through a lower cash flow volatility and a lower default probability in a higher firm value. Analysts should be aware of the benefit that diversified firms have through a decrease in default risk. With our model they are able to adopt this in the valuation process in a comfortable way. The results even point to the importance of explicitly modeling state-varying diversification across the business cycle. Since the diversification effect depends heavily on the economic state and is not constant across the business cycle, the implementation of copulas as a dependence structure enables analysts to incorporate this effect in valuation, e.g. the Clayton copula models higher dependence in times of crisis. Thus, asymmetric co-movements have implications for several applications. For example, in a firm's optimal portfolio allocation, if all businesses plunge down together as the economy falls, the value of diversification may be overstated by those not taking the increase in downside co-movements into account. These asymmetric co-movements of business segments should also be noticed when acquiring firms and new businesses. During an M&A-process the target business must be valued together with the businesses of the acquiring firm to cover the potential diversification effect through the risk reduction of the firm's portfolio. Research on how diversification can be implemented in firm valuation will thus be useful in a firms' future strategic planning while at the same time providing valuable information to investors and regulators as well as to rating agencies about default rates. Regulators can integrate the diversification effect in accounting standards to determine the fair value of subsidiaries in consolidated financial statements. However, in the wake of the global financial crisis of 2007 - 2009 it seems to be emerging that multi-business firms will continue to play an important role and state-varying diversification should be incorporated in their valuation.

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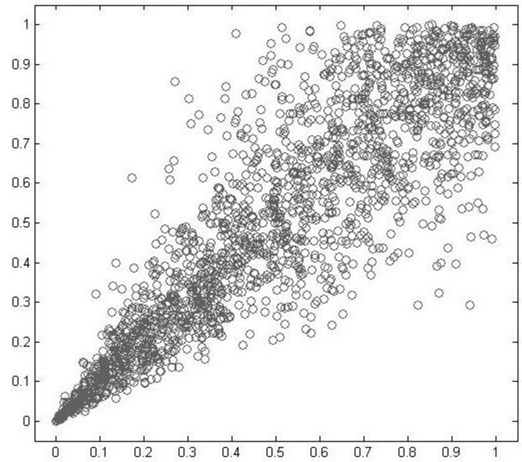
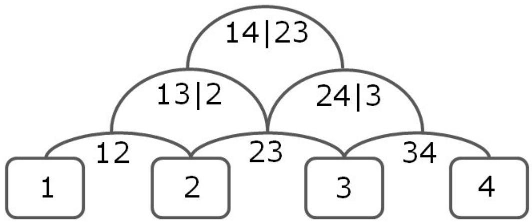
Figure 1: **Scatterplot of marginal distribution of Clayton copula**



The figure shows the asymmetric dependence structure with a lower tail dependence of a bivariate Clayton copula. The example is presented with a theta of 5.

Figure 2: **D-vine tree**



D-vine structure for four-dimensional pair-copula decomposition.

Table 1: **Firms and associated business segments**

Corporation	**No.**	**Business Segment**	**SIC Code**	**Industry Classification**
Servotronics	1	Advanced Technology Products	3621	22 Electrical Equipment
	2	Consumer Products	3421	17 Construction Materials
Sensient Technologies	1	Colors	2865	14 Chemicals
	2	Corporate & Other	2860	14 Chemicals
	3	Flavors & Fragrances	2869	14 Chemicals
Flowserve	1	All Other	4991	48 Other
	2	Flow Control	3491	17 Construction Materials
	3	Flow Solutions	3053	15 Rubber and Plastics
	4	Flowserve Pump	3561	21 Machinery

This table reports the three valuated firms with their associated business segments, segment SIC codes and Fama French 48 industry portfolio classification.

Table 2: **Input parameters of the model**

Item	**Notation**	**Estimation Procedure**
Initial revenues	$R_{l,0}$	Observable from current financial statement (FS)
Initial growth rate in revenues	$\mu_{l,0}$	Average log growth in revenues from past FS
Volatility of revenues	σ_l	Standard deviation of log growth in revenues from past FS
Long-term growth rate in revenues	$\overline{\mu}\ (= g)$	Equals expected average inflation rate g
Adjustment speed for the growth rate in revenues	κ	Estimated from the half-life of the process to $\overline{\mu}$
Segment weight	w_l	Average ratio of segment revenues to firm's total revenues from past FS
Initial operating working capital	$WC_{l,0}$	Firm's total non-cash working capital from current FS weighted with w_l
Initial net property, plant and equipment	$PPE_{l,0}$	Total net property, plant and equipment from current FS weighted with w_l
Ratio of capital expenditures to revenues	δ_l^{CE}	Average ratio of capital expenditures to revenues $R_{l,t}$ from past FS
Ratio of operating working capital to revenues	δ_l^{WC}	Average ratio of operating working capital to revenues $R_{l,t}$ from past FS
Ratio of depreciation/amortization to PPE	δ_l^{DE}	Average ratio of depreciation/amortization to $PPE_{l,t}$ from past FS
Fixed costs	F_l	Intercept of linear regression of costs on revenues
Variable costs as a percentage of revenues	$\tilde{\beta}_l$	Slope of linear regression of costs on revenues
Corporate tax rate	τ	Country-specific corporate tax rate
Cost of equity	r_{E_l}	Sum of five-year T-bond rate and Fama and French industry risk premium
Cost of debt	r_D	Average ratio of interest expenses to previous year's debt from past FS
Fixed proportion of debt in capital structure	w_D	Average ratio of debt to sum of equity and debt capital from past years
Fixed proportion of equity in capital structure	w_E	Average ratio of equity to sum of equity and debt capital from past years
Cash and marketable securities at valuation date	$CASH_0$	Observable from current FS
Debt at valuation date	$DEBT_0$	Observable from current FS
Minority holdings at valuation date	MH_0	Observable from current FS
Minority interests at valuation date	MI_0	Observable from current FS
Bankruptcy barrier	AF	Amount of potential additional financing
Parameter vector of Clayton copula	θ	Estimated from the log growth rates in revenues from past FS
Number of business segments	L	Differs from firm to firm (the sample contains firms with 2 to 4 segments)
Forecast horizon	T	Set to 25 years
Number of simulations	N	Set to 1,000,000
Discrete time interval	Δt	Set to 1 year

If the notation of a parameter includes the index l, it is to be considered segment-specific. The costs are computed segment-specific by taking revenues minus depreciation/amortization less operating profit. The market value of debt is approximated by the book value of interest bearing liabilities. The market value of equity is calculated from a firm's market capitalization observed from the stock market. The average ratios, average revenue growth rates, standard deviations and the linear regression of costs are calculated by data from the last 10 yearly financial statements. Note that the long term average growth rate $\overline{\mu}$ is chosen to be equal to the expected constant growth rate g. FS abbreviates financial statement.

Table 3: **Estimated input values of the three firms**

Segment data

	Servotronics		Sensient Technologies			Flowserve			
l	1	2	1	2	3	1	2	3	4
$R_{l,0}$	18.00	15.01	358.76	87.13	755.53	4.52	1,196.64	517.54	2,646.56
$\mu_{l,0}$	0.0608	0.0858	0.0415	0.0565	0.0394	-0.0407	0.1439	0.0200	0.2031
σ_l	0.1464	0.2476	0.1074	0.2592	0.0491	0.1231	0.3741	0.3414	0.2358
$\overline{\mu}$ $(= g)$	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03
κ	0.3466	0.3466	0.3466	0.3466	0.3466	0.3466	0.3466	0.3466	0.3466
w_l	0.5945	0.4055	0.3186	0.0581	0.6111	0.0025	0.2978	0.1841	0.5156
$WC_{l,0}$	7.95	5.43	136.91	24.95	262.62	0.98	115.23	71.21	199.49
$PPE_{l,0}$	3.75	2.56	135.60	24.71	260.10	1.43	166.92	103.16	288.97
δ_l^{CE}	0.0268	0.0267	0.0408	0.0932	0.0475	1.4343	0.0184	0.0213	0.0177
δ_l^{WC}	0.4605	0.4714	0.2621	0.2667	0.2613	0.1584	0.1712	0.1557	0.1589
δ_l^{DE}	0.1169	0.0644	0.0984	0.3360	0.1089	7.0094	0.1715	0.1279	0.1407
F_l	1.93	0.36	0.00	28.41	0.00	338.54	36.04	0.00	66.75
$\tilde{\beta}_l$	0.63	0.91	0.95	0.89	0.84	0.00	0.81	0.82	0.81
τ	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35	0.35
r_{E_l}	0.0936	0.1315	0.0875	0.0875	0.0875	0.0917	0.1315	0.1245	0.0923
r_D	0.0380	0.0380	0.0600	0.0600	0.0600	0.1177	0.1177	0.1177	0.1177
w_E	0.3327	0.3327	0.3293	0.3293	0.3293	0.3586	0.3586	0.3586	0.3586
w_D	0.6673	0.6673	0.6707	0.6707	0.6707	0.6414	0.6414	0.6414	0.6414
r_{wacc_l}	0.0707	0.0960	0.0716	0.0716	0.0716	0.0863	0.1118	0.1073	0.0867

Corporate data

	Servotronics	Sensient Technologies	Flowserve
$CASH_0$	4.32	12.22	654.32
$DEBT_0$	4.28	428.03	566.73
MH_0	0.00	0.00	0.00
MI_0	0.00	0.00	5.63
AF	1.65	60.07	218.26

The abbreviations of the segment data denote the following: l = segment number, $R_{l,0}$ = initial revenues, $\mu_{l,0}$ = initial growth rate in revenues, σ_l = volatility of revenues, $\overline{\mu}$ = long-term growth rate in revenues, κ = adjustment speed for the growth rate in revenues, w_l = segment weight, $WC_{l,0}$ = initial operating non-cash working capital, $PPE_{l,0}$ = initial net property, plant and equipment, δ_l^{CE} = ratio of capital expenditures to revenues, δ_l^{WC} = ratio of operating working capital to revenues, δ_l^{DE} = ratio of depreciation/amortization to PPE, F_l = fixed costs, $\tilde{\beta}_l$ = variable costs as a percentage of revenues, τ = U.S. corporate tax rate, r_{E_l} = cost of equity, r_D = cost of debt, w_E = fixed proportion of equity in capital structure, w_D = fixed proportion of debt in capital structure, and r_{wacc_l} = weighted average cost of capital. The abbreviations of the data for the entire firm denote the following: $CASH_0$ = cash and marketable securities, $DEBT_0$ = interest bearing liabilities, MH_0 = minority holdings, MI_0 = minority interests, and AF = bankruptcy barrier. If F_l or $\tilde{\beta}_l$ are negative, we set them to zero. All values are measured in million US$, except percentages.

Table 4: **Clayton copula parameter**

Corp.	$\theta_{1,2}$	$\theta_{2,3}$	$\theta_{3,4}$	$\theta_{13\|2}$	$\theta_{24\|3}$	$\theta_{14\|23}$
Servotronics	1.7843					
Sensient Technologies	1.4983	0.0002		1.1721		
Flowserve	1.6844	0.0002	0.4240	0.4949	0.6502	0.0853

This table reports the estimated Clayton copula parameters by optimizing the copula vine log-likelihood (see equation (26)) as the sum of $L(L-1)/2$ bivariate likelihoods. The structure of the D-vine tree for the four-dimensional pair-copula decomposition (e.g. Flowserve) is shown in figure 2.

Table 5: **Simulated intrinsic equity values and cumulative default probabilities**

Servotronics	**Scenario A** Independence		**Scenario B** Copula dependence		**Scenario C** Perfect dependence
Equity Value	45.18		34.49		18.88
Relative difference		-31.00%		-45.24%	
10-year cumulative p.d.	36.70%		41.61%		49.51%
Absolute difference		4.91%		7.90%	
25-year cumulative p.d.	50.89%		54.01%		64.54%
Absolute difference		3.12%		10.53%	

Sensient Technologies	**Scenario A** Independence		**Scenario B** Copula dependence		**Scenario C** Perfect dependence
Equity Value	495.89		479.26		375.41
Relative difference		-3.47%		-21.67%	
10-year cumulative p.d.	0.43%		0.94%		4.99%
Absolute difference		0.51%		4.06%	
25-year cumulative p.d.	4.91%		7.13%		15.96%
Absolute difference		2.23%		8.82%	

Flowserve	**Scenario A** Independence		**Scenario B** Copula dependence		**Scenario C** Perfect dependence
Equity Value	5,722.64		5,361.58		4,763.15
Relative difference		-6.73%		-11.16%	
10-year cumulative p.d.	17.07%		20.50%		24.60%
Absolute difference		3.42%		4.11%	
25-year cumulative p.d.	17.77%		22.30%		29.47%
Absolute difference		4.53%		7.17%	

This table presents the simulation results of the three diversification scenarios for the three U.S. firms. The equity values are given in million U.S. $ and the cumulative probabilities of default (cumulative p.d.) are given in percent. The differences between scenario A and B as well as the differences between scenario B and C are also presented. The differences between the equity values are relative differences (scaled by equity value of scenario B) whereas the differences between the cumulative probabilities of default are absolute.